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LETTER TO THE EDITOR

Conjectured singularity loci for the Potts model on the square lattice†

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Abstract. It is conjectured that the singularities in the free energy for the q-state Potts model on the square lattice lie on two self-dual circles in the complex x plane where $x = \exp(-J/kT)$ is the usual low-temperature variable. These circles are natural generalisations of the known circles for the Ising model ($q = 2$). The conjecture leads to the prediction that there is a single antiferromagnetic critical point at $x = 1 + [q/(q-1)]^{1/2}$.

The Potts (1952) model has attracted much attention in recent years (Mittag and Stephen 1971, Straley and Fisher 1973, Baxter 1973, Zwanzig and Ramshaw 1977, Schick and Griffiths 1977, Hintermann *et al* 1978). The Potts model is a generalised Ising model in which the sites of a regular lattice are occupied by 'spins', each of which may exist in any of q distinct states. The interaction energy of a pair of spins is zero unless they are nearest neighbours in the same state, in which case it is $-J$. Instead of the temperature T, it is convenient to utilise the independent variable $x = \exp(-J/kT)$, where k is Boltzmann's constant.

We present here a conjecture as to the loci, in the complex x plane, of the singularities in the free energy for the Potts model on the square lattice. The conjecture leads to the prediction that there is a single antiferromagnetic critical point at $x = 1 + [q/(q-1)]^{1/2}$.

Our conjecture is based upon the duality theorem (Potts 1952, Kihara *et al* 1954, Mittag and Stephen 1971) and the known behaviour of the case $q = 2$ (Ising model). Let $A(x) = -F/kT$, where F is the free energy per lattice site. The duality theorem may be stated in the form (Zwanzig and Ramshaw 1977)

$$A(x) - \ln[1 + (q-1)x^2] = A(t) - \ln[1 + (q-1)t^2], \tag{1}$$

where x and t are related by the duality transformation

$$t = (1-x)/[1 + (q-1)x], \qquad x = (1-t)/[1 + (q-1)t]. \tag{2}$$

Clearly, if $A(x)$ is singular at the point $x = x_0$, then it will also be singular at the dual point $x = (1-x_0)/[1 + (q-1)x_0]$.

Keeping the duality theorem in mind, let us review the known behaviour of the case $q = 2$. For this case, the singularities in $A(x)$ lie on the two circles (Fisher 1965, Brascamp and Kunz 1974)

$$x = \mp 1 + \sqrt{2}\exp(\mathrm{i}\phi) \qquad (0 \leq \phi < 2\pi), \tag{3}$$

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which will be referred to as the left and right circles respectively. It is of interest to see how the duality relation between singular points is satisfied for these circles. One readily verifies that both circles are self-dual; i.e. the dual to any point on either circle is another point on the same circle. However, the two circles are self-dual in quite different ways. On the left circle, the dual to a point x is just the conjugate point x^*; i.e. $t(x)=x^*$. On the right circle, the straight line connecting a point x to its dual always passes through the origin, so that $t(x)=-ax$, where a is real and positive (and depends on ϕ).

A physical singularity can occur only when one of the circles intersects the real axis within the physically permissible range of x. For $J>0$ (ferromagnetic case) the physical range of x is $0 \leqslant x < 1$. The only intersection in this range is at the point $x_f = \sqrt{2}-1$, which is the known ferromagnetic critical point. For $J<0$ (antiferromagnetic case) the physical range of x is $1 < x < \infty$. The only intersection in this range is at the point $x_a = \sqrt{2}+1$, which is the known antiferromagnetic critical point.

Unphysical singularities are also of interest, since they can interfere with attempts to extract information about the physical singularity from power series expansions. For $q=2$, it is known that $A(x)$ contains the term $\ln(1+x^2)$, which is highly singular at the points $x = \pm i$ at which the two circles intersect. It is of interest to interpret this term with reference to the roots of the partition function Q_N for a system of N spins (N large). The quantity $A(x)$ is related to Q_N by $A(x) = \lim_{N\to\infty}(N^{-1} \ln Q_N)$. Now Q_N is a polynomial in x of degree $2N$, and it has real coefficients. It follows that Q_N has $2N$ roots, and that they occur in conjugate pairs. Since

$$N^{-1} \ln[(x-i)^N (x+i)^N] = \ln(1+x^2), \tag{4}$$

the presence of the term $\ln(1+x^2)$ in $A(x)$ implies that, for large N, almost all of the $2N$ roots of Q_N converge to the points $x = \pm i$ at which the two circles intersect.

We now observe that circles with similar duality properties can be found for arbitrary q. The condition $t(x)=x^*$ implies that x lies on the self-dual circle

$$x = -1/(q-1) + q^{1/2} \exp(i\phi)/(q-1) \qquad (0 \leqslant \phi < 2\pi), \tag{5}$$

which is the generalisation of the left circle to arbitrary q. The condition $t(x) = -ax$, where a is real and positive, implies that x lies either on the real axis (which is not of interest) or on the self-dual circle

$$x = 1 + [q/(q-1)]^{1/2} \exp(i\phi) \qquad (0 \leqslant \phi < 2\pi), \tag{6}$$

which is the generalisation of the right circle to arbitrary q. Our conjecture is that the singularities in $A(x)$ lie on the circles (5) and (6) for arbitrary q.

With regard to physical singularities, the conjecture implies the following. The only intersection of a circle with the real axis in the ferromagnetic range $0 \leqslant x < 1$ occurs at the point $x_f = (q^{1/2}-1)/(q-1)$, which is the ferromagnetic critical point found by Potts (1952). The only intersection of a circle with the real axis in the antiferromagnetic range $1 < x < \infty$ occurs at the point $x_a = 1 + [q/(q-1)]^{1/2}$, which is therefore the antiferromagnetic critical point.

The point x_f is fortuitously self-dual, a property which makes it conspicuous and which led to its early discovery. The point x_a is not self-dual; it is therefore less conspicuous, which is the reason it has heretofore escaped detection for $q > 2$.

With regard to unphysical singularities, it is natural to assume that, for large N, almost all of the $2N$ roots of Q_N continue to converge to the points at which the two circles intersect. For arbitrary q these intersections occur at the points $x = +i(q-1)^{1/2}$.

This assumption implies the presence of a term $\ln[1+(q-1)x^2]$ in $A(x)$, and suggests that the quantity

$$B(x)=A(x)-\ln[1+(q-1)x^2], \tag{7}$$

which incidentally is duality-invariant according to equation (1), will be free of these unphysical singularities and hence more susceptible to analysis. $B(x)$ is precisely the quantity analysed by Zwanzig and Ramshaw (1977), which perhaps explains why their analysis worked so well. (The mere fact that $B(x)$ is duality-invariant is insufficient explanation, since $A(x)$ may be separated into a duality-invariant part and a remainder in an infinite number of ways.)

The antiferromagnetic ($J<0$) behaviour of the Potts model has received surprisingly little attention. The only such work of which we are aware is that of Schick and Griffiths (1977), who studied the case $q=3$ on the triangular lattice using a renormalisation group technique. It is to be hoped that further studies will be forthcoming, and that they will lead to a rigorous determination of x_a.

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